A Practical View of Rotations in U = DH Systems

1. Setup

We study the iteration

$x_{t+1} = Ux_t, \quad U = DH$

where:

- H is the (normalized) Walsh–Hadamard transform
- D = diag(±1)

Both are reflections. Their product produces rotations in hidden 2D planes.

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2. The key simplification: two big pieces

The crucial property of H:

- eigenvalues are +1 and -1
- so every vector splits into two parts

You can compute this split directly:

$x_+ = \frac{1}{2}(x+Hx), \quad x_- = \frac{1}{2}(x-Hx)$

These satisfy:

- $Hx_+ = +x_+$
- $Hx_- = -x_-$

So at any time:

$x_t = x_{t+} + x_{t-}$

Interpretation:

- x_+: “even / aligned with H”
- x_-: “odd / flipped by H”

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3. What the system actually does

Each step:

1. Apply H: flips the x_- part
2. Apply D: flips selected coordinates

The important effect:

> D mixes the (+) and (−) parts.

So over time:

- energy moves from x_+ to x_- and back
- this exchange creates rotation

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4. Where the rotations come from

Instead of thinking about many abstract planes, use this:

> Every rotation comes from a **pairing between a direction in x_+** and a **direction in x_-**.

Inside such a pair:

- one component flips under H
- the other doesn’t
- D couples them

That pair behaves like:

$$(a_t, b_t) \mapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} (a_t, b_t)$$

So the system is:

- many such pairings

- each rotating at its own speed

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5. What you can *measure directly*

(A) Energy split between + and −

(A) Energy split between $+$ and $-$

Compute at each step:

$$E_+(t) = \|x_{t,+}\|^2, \quad E_-(t) = \|x_{t,-}\|^2$$

Observations:

- $E_+(t) + E_-(t)$ is constant
- energy oscillates between them
- oscillation frequencies = rotation speeds

This is the simplest “global” observable.

(B) Track a single coordinate

Pick index i, record $x_t[i]$.

You’ll see:

- quasi-periodic oscillation

Apply a Fast Fourier transform:

- peaks correspond to rotation angles Θ_k

(C) Two-coordinate plots

Plot:

(xt[i], xt[j])

You'll see:

- loops / Lissajous figures
- mixtures of multiple rotations

Occasionally:

- near-ellipses (dominant plane)

(D) Direct "sloshing" view

Track:

$$s_t = \langle x_{t,+}, x_{t,-} \rangle$$

This measures coupling between the two halves.

Typical behavior:

- oscillatory
- tied to rotation phases

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6. Initial state: what gets excited

Single spike

Let:

$x_0 = e_i$

Then:

- spreads into both x_+ and x_-
- excites many rotation modes

Practical takeaway:

> a single nonzero entry gives you a full "spectrum probe" of the system

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Multiple inputs

Because the system is linear:

$$U^t(a+b) = U^t a + U^t b$$

So:

- trajectories superpose
- no interaction between modes

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7. What controls the rotation speeds

All angles Θ_k come from how D mixes the two halves:

- if D barely mixes x_+ and x_- → slow rotations
- strong mixing → faster rotations

Heuristically:

- H defines the split
- D defines the coupling

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8. A useful mental model

Think of:

- x_+ and x_- as two “reservoirs”
- D as a valve moving energy between them

Each independent exchange channel:

- becomes a rotation mode

So the whole system is:

> many independent energy exchanges, each oscillating at its own rate

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9. Minimal workflow for exploration

1. Choose D
2. Initialize $x_0 = e_i$
3. Iterate $x_{t+1} = DHx_t$
4. At each step:
    - compute x_{t+}, x_{t-}
    - log E+(t), E-(t)
    - record a few coordinates
5. Analyze:
    - FFT → rotation speeds
    - plots → geometric intuition

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10. Key takeaway

> The rotations in U = DH arise because H splits space into two parts, and D continuously mixes them.

Everything you observe—oscillations, patterns, “complexity”—is just:

- energy moving between x_+ and x_-
- decomposed into many independent circular motions.

Appendix: Full solution via eigenvectors (brief)

For completeness, the system

$$x_{t+1} = Ux_t, \quad U = DH$$

can be solved exactly using standard methods from linear algebra.

- Compute the eigenvalues and eigenvectors of U:

$$Uv_k = \lambda_k v_k, \quad \lambda_k = e^{i\theta_k}$$

- For each complex eigenpair:
  - take real and imaginary parts of v_k
  - these span one of the 2D rotation planes
- The general solution is:

$$x_t = \sum_k c_k \lambda_k^t v_k$$

In real terms, this is equivalent to:

- decomposing the state into invariant 2D planes
- each evolving as a pure rotation with angle θ_k

Practical note:
While this gives a complete description, it is often less intuitive and more computationally involved than the $x_+ + x_-$ splitting and observational methods described in the main text.